Phase transition analysis in the quantum control landscape

Nicolò Beato, Pranay Patil, Marin Bukov
Max Planck Institute for the Physics of Complex Systems, Dresden, Germany

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Talk overview

1. The quantum control problem
2. Phase transition in the single qubit control landscape

The quantum control problem

1. Time-dependent hamiltonian:

$$\hat{H}(t) := h_z \hat{S}_z + s(t) h_x \hat{S}_x$$

"Protocol" $= s(t) \in \{\pm 1\}$

2. Discretize time axis:

$$[0, T] \mapsto \{t_1, t_2, \ldots, t_{N_T}\}$$

$$s(t) \mapsto \{s_1, s_2, \ldots, s_{N_T}\}$$

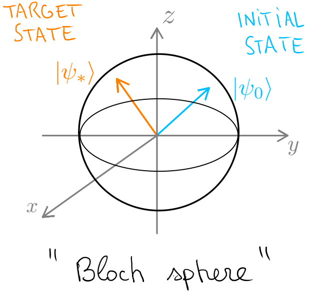
3. Evolution operator:

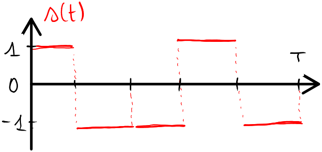
$$\hat{U}_s(T) = \hat{U}_{s_{N_T}} \circ \cdots \circ \hat{U}_{s_2} \circ \hat{U}_{s_1}$$

4. Goodness of the protocol:

$$F_s(T) := \left| \langle \psi_* | \, \hat{U}_s(T) \, | \psi_0 \rangle \right|^2 \in [0, 1]$$

called "Fidelity"

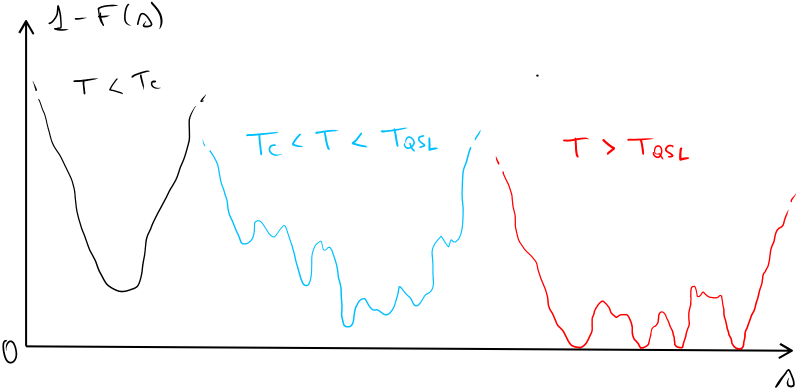




The quantum control problem

The quantum control problem:

1. $s(t)$ "optimal" if $F_s(T) = 1$
2. How many optimal protocols are there?
3. How difficult is the search for optimal protocols?



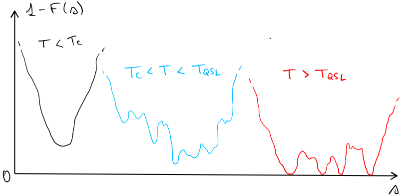
Phase transition in the quantum control landscape

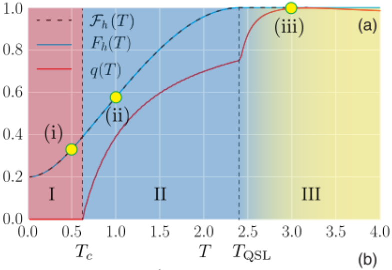
Previously:

1. Sampled optimal protocols using numerical methods
2. Properties of the landscape captured by the order parameter

$$q(T) := 1 - \frac{1}{N_T} \sum_{i=1}^{N_T} \overline{s_i}^2$$

$\overline{s_i}$:= (average over optimal protocols)

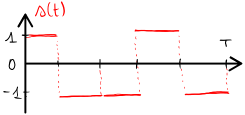


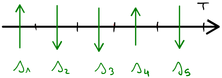


Mapping to a classical spin model

The fidelity can be expanded in the general form

$$F_s(T) = a(T) + \sum_{i=1}^{N_T} b(T)_i s_i + \sum_{i<j}^{N_T} c(T)_{ij} s_i s_j + \sum_{i<j<k}^{N_T} d(T)_{ijk} s_i s_j s_k + \ldots$$





1. $-\log(F_s(T)) =$ (energy of the classical N_T-spin model) $\in [0, +\infty)$
2. "Ground states" of the classical spin model = "optimal protocols" of quantum control problem
3. Idea: use knowledge from statistical mechanics of classical spin models to deduce properties about the quantum control problem

My project: In order to study the classical spin model we first need to compute the coupling $a(T), b(T)_i, c(T)_{ij}, \ldots$

Compute couplings

$$F_s(T) = \left| \langle\psi_*| \hat{U}_s(T) |\psi_0\rangle \right|^2 = \langle\psi_*| \hat{U}_s(T) |\psi_0\rangle\langle\psi_0| \hat{U}_s(T)^\dagger |\psi_*\rangle .$$

1. Since $|\psi_0\rangle\langle\psi_0| \leftrightarrow \vec{n}_0 \in$ (Bloch sphere),

$$\hat{U}(\vec{s}, T) |\psi_0\rangle \langle\psi_0| \hat{U}(\vec{s}, T)^\dagger \leftrightarrow M(\vec{s}, T)(\vec{n}_0),$$

2. In discrete time we have

$$M_s(T) = M_{s_{N_T}} \circ \cdots \circ M_{s_2} \circ M_{s_1}$$

3. Decomposing $M_{s_i} = A + s_i B$, eventually

$$M_s(T) = \left[I + \sum_{i=1}^{N_T} O_i s_i + \sum_{i<j}^{N_T} O_i O_j s_i s_j + \sum_{i<j<k}^{N_T} O_i O_j O_k s_i s_j s_k + \dots \right] A^{N_T}$$

where $O_i := A^{i-1}(BA^{-1})A^{-(i-1)}$

Compute couplings, $N_T \to \infty$

$$M_s(T) = \left[I + \sum_{i=1}^{N_T} O_i s_i + \sum_{i<j}^{N_T} O_i O_j s_i s_j + \sum_{i<j<k}^{N_T} O_i O_j O_k s_i s_j s_k + \dots \right] A^{N_T}$$

$$\overset{N_T \to \infty}{\to} \left[I + \int_0^T O(t)s(t)dt + \iint_{t_1<t_2}^T O(t_1)O(t_2)s(t_1)s(t_2)d^2t + \dots \right] A^{N_T}$$

$=$"Dyson's series".

So, compact form:

$$M_s(T) = \mathcal{T}\left[\exp\left(\int_0^T O(t)s(t)dt \right) \right] A^{N_T}.$$

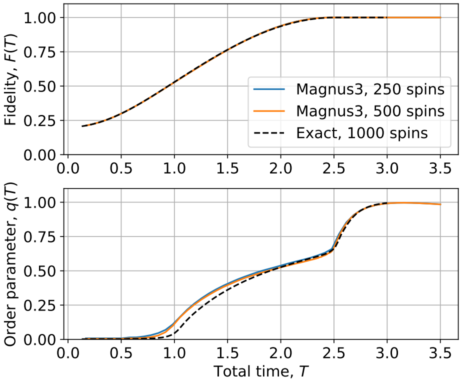
Also, "Magnus's expansion" of the time-ordered evolution operator:

$$M_s(T) = \exp\left(\int_0^T O(t)s(t)dt + \iint_{t_1<t_2}^T [O(t_1), O(t_2)] s(t_1)s(t_2)d^2t + \dots \right) A^{N_T}$$

Conclusion

- Two different functional power expansion for the $F_s(T)$: Dyson and Magnus
- Truncating the power expansion gives an approximated but analytical model

From stochastic descent:



Summary

So far:

- Analytic functional approximation of fidelity for the single qubit system

Next:

- Extract information about the quantum control landscape phase transition
  - Critical exponents
  - Universality class?

Outlook:

- More complicated systems: $SU(3)$, two-qubit system, ...
- Multiple control parameters: $s_1(t), s_2(t), \ldots$

Thank you!